Lecture 14

Perturbations in Inflation

- Review
- Free scalar field in Minkowski
- Free scalar field in de Sitter
- Inflationary perturbations

GR 13.1

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- Review
  - → Primordial perturbations → CMB
  - → $\varphi \underset{t\to\infty}{\simeq} 6\,\varphi_i \cos u_s k \eta$

$$\langle \varphi^i_{k_1} \varphi^i_{k_2} \rangle = \delta(\vec{k_1} - \vec{k_2})\, P_{|k|}$$

$$P(k) \sim 10^{-10} \left(\frac{k_*}{k}\right)^{3+(n_s-1)}$$

→ Inflation at the background level:

$$\varepsilon = \frac{M_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \qquad \eta = \frac{M_{pl}^2}{8\pi}\frac{V''}{V}$$

take $V = \frac{m^2}{2}\varphi^2$, conditions:

$$\frac{M_{pl}^2}{\varphi^2} \ll 1 \qquad \frac{M_{pl}^2}{\varphi^2} \ll 1$$

$$m^2 \varphi^2 \ll M_{pl}^4$$

- We will show that quantum mechanics creates classical perturbations that naturally have the properties required by observations. Essentially, these are ground-state fluctuations of a (free) massless scalar field.

Free scalar field in Minkowski

$$S_\varphi = -\frac{1}{2}\int d^4x\, \eta^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi$$

$$E = H = \frac{1}{2}\int d^3x\left(\dot{\varphi}^2 + (\partial_i\varphi)^2\right)$$

This is just an infinite set of harmonic oscillators

$$\varphi(x,t) = \int \frac{d^3q}{(2\pi)^{3/2}\sqrt{2\omega_q}} \left(e^{i\omega_q t - i\vec{q}\cdot\vec{x}} A_q^+ + e^{-i\omega_q t + i\vec{q}\cdot\vec{x}} A_q \right)$$

$$\omega_q = |\vec{q}|$$

$$[A_q, A_{q'}^+] = \delta^3(q - q')$$

$$H = \int d^3q \, \omega_q A_q^+ A_q$$

e.g. harmonic oscillator

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 x^2 \qquad x \simeq \frac{a + a^+}{\sqrt{\omega}}$$

$$a = \sqrt{\frac{m\omega}{2}} \left(x + \frac{iP}{m\omega} \right) \qquad a^+ = \sqrt{\frac{m\omega}{2}} \left(x - \frac{iP}{m\omega} \right)$$

$$H = \omega a a^+ \qquad [a, a^+] = 1$$

As is well known, in the ground state

$$\langle x \rangle = 0 \qquad \langle x^2 \rangle = \frac{1}{m\omega}$$

$$\langle \varphi_q \varphi_{q'} \rangle = \delta^3(q+q')\frac{1}{\omega} = \delta^3(q+q')\frac{1}{|q|}$$

[since we often need to integrate $\int d^3q\, P(q)$ often $q^3 P_q$ is used]

Inflationary perturbations

- In order to develope the full theory of inflationary perturbations we need to

$\rightsquigarrow$ repeat the steps of classical theory of perturbations (fix the gauge, S-V-T decomposition, etc. for inflationary theory)

$\rightsquigarrow$ quantize the corresponding fields

- We will do a simplified treatment and ignore the metric perturbations, quantizing just the perturbations of the inflaton field.
- Turns out, for scalar perturbations it gives the right answer to leading order in slow-roll parameters (ε, η)
- There is no conceptual difficulty in putting together the two steps, but technically is a bit complicated

- Inflaton field perturbations:

$$\phi(x,t) = \varphi_{cl}(t) + \varphi$$

small $\sim \eta$

$$S_\varphi = \frac{1}{2}\int d^4x \sqrt{-g}\left(-g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - \cancel{V''(\varphi_{cl})}\,\varphi^2\right)$$

(We did it at the level of EOM, but can also be done for the action)

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{a^2}\partial_i\partial_i\varphi \quad \text{in FRW time}$$

or in conformal time

$$\varphi'' + 2\frac{a'}{a}\varphi' - \Delta\varphi = 0$$

$$\frac{a'}{a} \sim \frac{1}{|\eta|}, \quad \varphi' \sim k, \quad \Delta\varphi \sim k^2$$

$\frac{1}{|\eta|}k \gg k^2$ $\quad |\eta k| \ll 1 \to$ outside horizon

$|\eta k| \gg 1 \to$ inside horizon

$t \nearrow \Rightarrow |\eta| \searrow$

$t \to -\infty \Rightarrow |\eta| \to \infty \Rightarrow$

$$\varphi'' - \Delta\varphi = 0 \Rightarrow$$

same as

in flat space ~ expect ground state oscillations.

define

$$\chi = a(\eta)\,\varphi \,, \quad \text{then}$$

$$S_\chi = \frac{1}{2}\int d^3x\, d\eta \left[\chi'^2 - (\partial_i \chi)^2 + \frac{a''}{a}\chi^2 \right]$$

harmonic oscillators with time-dependent frequencies!

Fourier transform and quantize:

$$\chi(x,\eta) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2k}} \left(e^{-ikx}\, \chi_k^{+}(\eta) A_k^{+} + e^{i\vec{k}\vec{x}}\, \chi_k^{(-)}(\eta)\, A_k \right)$$

$$[A_k, A^\dagger_{k'}] = \delta(\vec{k} - \vec{k}')$$

what are the functions $\chi^\pm(\eta)$?

in flat space we had $e^{\pm i\omega\eta}$.

it's just solution of EOM: $\ddot{\varphi} = \omega^2\varphi$

• Let's use de Sitter approximation:

$$a(\eta) = \frac{1}{|\eta| H}$$

Then classical EOM for χ_k is

$$\chi_k''^{cl} - \frac{2}{\eta^2}\chi_k^{cl} + k^2\chi_k^{cl} = 0$$

Quantum operator $\hat{\chi}_k$ satisfies the same equation (its just time-dependent-frequency harmonic oscillator)

$\left[\dot{p} = [H, x] = \omega^2 x \right.$, $\left. \dot{x} = [H, p] = p \right]$

$$\chi^{\pm} = e^{\pm i k \eta}\left(1 \pm \frac{i}{k\eta}\right)$$

two solutions of classical EOM,
we pick them by matching to
flat space results in $k\eta \to \infty$ limit
(very similar logic to Lecture 12 in
the classical case)

Now, in the super-horizon region
we get

$$\langle \varphi(k)\, \varphi(k') \rangle \sim \delta^3(k-k') \cdot$$

$$\cdot a^{-2}(\eta) \frac{1}{|k\eta|^2} \cdot \frac{1}{k} =$$

$$= \delta^3(k-k') \frac{H^2}{k^3} \qquad k\eta \ll 1$$

$$\text{vs} \quad \delta^3(k-k') \frac{1}{k} \quad \text{in flat space}$$

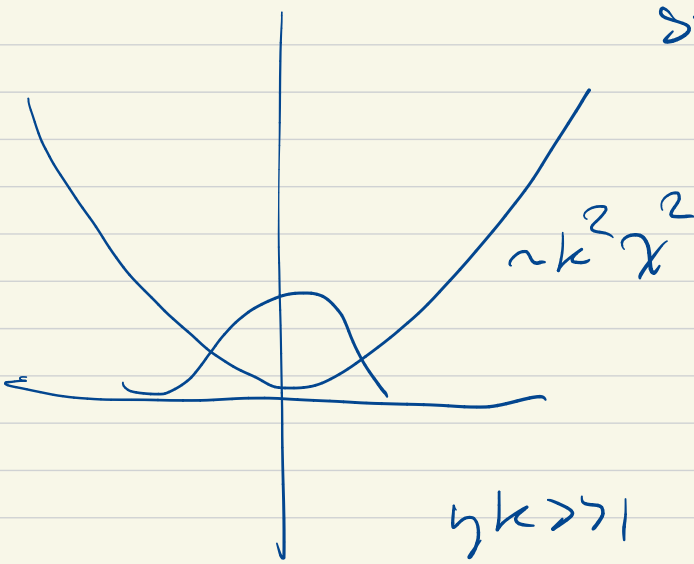
This is the famous scale-invariant power-spectrum!

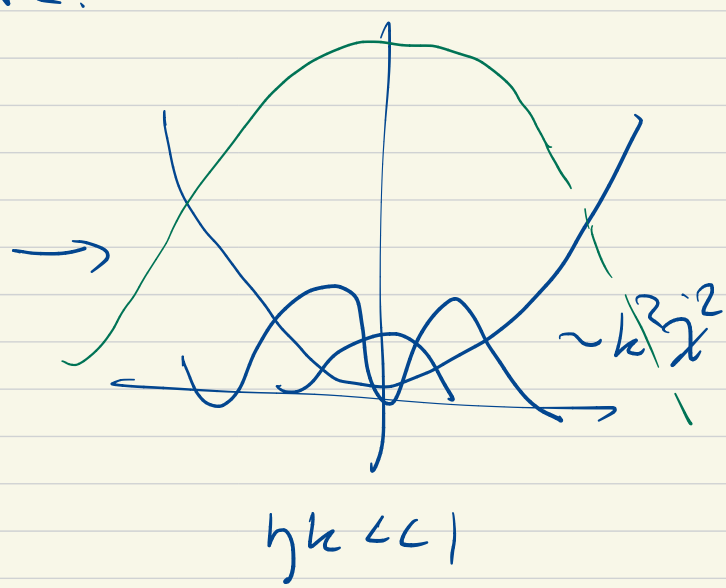
$$d^3k \, \frac{1}{k^3} \simeq \text{const}, \qquad \langle \varphi(x)\varphi(y) \rangle \sim \log(x-y)$$

- So far we computed quantum perturbations of $\delta\varphi$. (We did an "exact" QM computation). In fact, superhorizon modes are classical.

$$\frac{1}{|k\eta|^2}\frac{1}{k} \gg \frac{1}{k} \quad \sim \quad \chi_k \text{ oscillator is}$$

in a very highly occupied state:

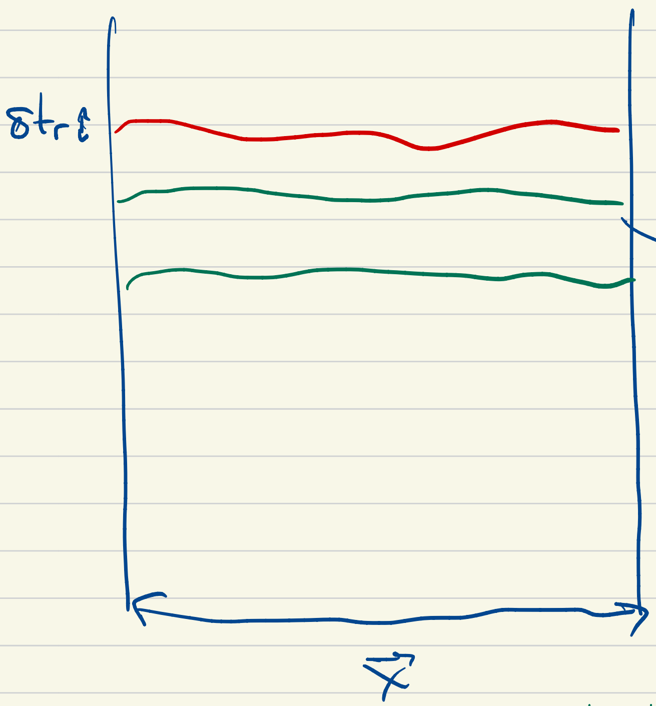




High occupation numbers $\sim$ classical state.

- We derived the spectrum of $\varphi \equiv \delta\phi$, how do we transform it into $\Phi_i(k)$ — ?

- Inflation always ends when $\phi = \phi_{cl} + \varphi = \phi_r$ this happens at different times in different parts of the universe:



$\uparrow t$ $\qquad \phi = \phi_r$

lines of const. ϕ

$$\delta t_r(\vec{x}) = \frac{\varphi(\vec{x})}{\dot{\phi}_{cl}}$$

$$a = e^{Ht} \Rightarrow$$

gr. potential

$$\left.\frac{\delta a}{a}\right|_{t_r} \simeq \Phi_i \simeq \frac{H\,\varphi(\vec{x})}{\dot{\phi}_{cl}}$$

inflaton (not same letter)

$$\phi_{cl} = \sqrt{V \varepsilon} \simeq H M_{pl} \sqrt{\varepsilon}$$

φ_i is a classical "random" variable which expectation value is determined by QM (as always in the measurement)

$$\underbrace{\langle \varphi_i \varphi_i \rangle}_{\text{average over } k} = \frac{H^2}{\varepsilon H^2 M_{pl}^2} \langle \varphi \varphi \rangle =$$

$$= \frac{H^2}{\varepsilon M_{pl}^2} \frac{1}{k^3} \delta^3(\vec{k} - \vec{k}')$$

$$\frac{H^2}{\varepsilon M_{pl}^2} \simeq 10^{-10} \quad \text{to match observations}$$

$H \ll M_{pl} \rightarrow$ good, quantum gravity can be ignored

But scale of inflation is not yet fixed.

- Importantly, we have a constraint on ε and η: slow-roll corrections produce a tilt in the power spectrum: $H = H(t)$:

$$P(k) \simeq \frac{1}{k^{3+1-n_s}}$$

$$n_s - 1 = 2\eta - 6\varepsilon$$

$$n_s = 0.965 \pm 0.004 \quad < 1 \, !$$

natural $\varepsilon, \eta \sim 10^{-2}$: slow-roll approximation is good

- We did not talk about
  - → Non-gaussianities
  - → tensor modes (gravity waves from inflation)